PERGAMON

International Journal of Solids and Structures 37 (2000) 1119–1132

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The Reissner–Sagoci problem for a half-space under buried torsional forces

M. Rahman[1]

3019 W. 13 Mile Rd., Apt. 214, Royal Oak, MI 48073, USA

Received 1 November 1997; in revised form 24 August 1998

Abstract

The article extends Reissner and Sagoci's classical solution to the problem of a rigid circular punch bonded to a homogeneous, elastic isotropic half-space in which there is an axisymmetrical distribution of buried torsional forces. The surface of the half-space is free from stresses. The punch undergoes rotation due to the action of the internal loads. Solution of the problem is obtained by superposing the solutions of two simpler problems, viz the problem of the elastic half-space without the punch under the action of the prescribed torsional forces and the contact problem for the half-space with the rigid circular punch bonded to its surface, which is subjected to some tangential displacement. The form of this tangential displacement is determined from the solution of the first problem. Exact solutions of both problems are derived by constructing the Green's function, which corresponds to the action of a unit concentrated force uniformly distributed along a circular ring in the tangential direction. Specific examples are considered. Furthermore, an extension of these results to the case of a transversely isotopic half-space is presented. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Problems concerning contact between deformable solids are of considerable theoretical and practical importance, since contact is the commonest way to transmit loads from one structural member to another. This is why contact mechanics continues to be one of the most important branches of theoretical elasticity. Significant achievements of both theoretical and computational nature have been made in this area since the time of Heinrich Hertz. Extensive account of this progress is given by Ufliand (1965), Galin (1976), De Pater and Kalker (1975), Gladwell (1980), Johnson (1985) and Kikuchi

E-mail address: mrahman10@hotmail.com (M. Rahman).

[1] I dedicate the paper to the fond memory of my mother Rowshonara Begum whose constant love and encouragement were crucial for my education and self-development.

PII: S0020-7683(98)00277-7

and Oden (1988). However in these treatments, it is explicitly assumed that the contact between deformable solids is achieved by directly loaded punches. But there is another class of contact problems where the contact between solids is perturbed by forces that are applied in the interior of the medium. Contact problems of this category have received only limited attention. But they are of considerable practical interest in many areas, such as geomechanical applications wherein the internal forces can be visualized as forces transmitted by anchoring regions located in the vicinity of structural foundations (Selvadurai, 1981). Also, as discussed by Selvadurai (1981), in the particular case where the internal force migrates to the surface of the half-space, the resulting problem models the interaction between the existing structural foundation and a surcharge load applied at its vicinity. Recent interest in this class of problems is due to a number of papers by Selvadurai (1978, 1990) and Fabrikant et al. (1985).

The present article investigates the problem of a rigid circular punch bonded to the surface of a homogeneous, elastic isotropic half-space loaded internally by torsional forces. The corresponding problem where contact is achieved by a directly loaded indenter was considered by a number of authors. The classical results in this area belong to Reissner and Sagoci (1942) who used oblate spheroidal coordinates to solve the problem. Sneddon (1944, 1951) solved the statical counterpart of the problem by an approach based on Hankel transforms and dual integral equations. Excellent review of the later work can be found in Ufliand (1965), De Pater and Kalker (1975), Galin (1976), Gladwell (1980), Johnson (1985) and Hills et al. (1993). Of the most recent work in this area, mention may be made of the paper by Hanson and Puja (1997), where an extension to these classical results is given for a transversely isotropic half-space.

Within the scope of linear elasticity, the solution of the titled problem is found by superposition of the solutions of two simpler problems. The first problem consists in finding the elastic field in the half-space without the punch under the action of the prescribed internal loading while the second problem aims at finding a corrective solution of the problem of the elastic half-space with the bonded punch in which the punch is subjected to some tangential displacement. The form of this tangential displacement is determined from the solution of the unperturbed problem. To solve both problems, we develop a Green's function, which consists in finding the elastic field in the half-space due to the action of a unit concentrated load uniformly distributed along a circular ring in the tangential direction, in a plane parallel to the surface of the half-space. The solution of the unperturbed problem is then found by integrating the Green's function with the prescribed internal loading as the weight over the whole half-space region. Using the same Green's function, the perturbation problem is then reduced to an integral equation with the contact stresses under the punch as the unknown quantity. The solution of the integral equation is obtained in closed form. Specific cases of internal loading are considered. It is worth mentioning in this context that the titled problem for a transversely isotropic half-space was solved by Selvadurai (1982). However, his solution is restricted to the consideration of the case where the internal loading is effected by a concentrated couple only and it is not clear how to generalize his results. The author believes that it is, in fact, not possible to adapt his results to generate solution for the case where the internal loading is effected by any arbitrary axisymmetrical distribution of torsional forces. In this sense, the approach developed in the present article is more general. Specific examples are considered to demonstrate the generality of the present approach.

We begin by introducing the notation that we shall make use of.

We define the Hankel transform of order $v (v \geqslant -\frac{1}{2})$ of a function $f(r)$ by the equation (Sneddon, 1972)

$$\widetilde{f}^{v}(s) = \int_0^{\infty} r f(r) J_v(rs)\, \mathrm{d}r,$$

where $J_v(rs)$ is the Bessel function of the first kind and of order v. We write the above relation as

$\widetilde{f}^{v}(s) = \mathscr{H}_v[f(r); r \rightarrow s]$. The inversion theorem for the Hankel operator states that if $\widetilde{f}^{v}(s)$ is the Hankel transform of order v of the function $f(r)$, then

$$f(r) = \int_0^\infty s\widetilde{f}^{v}(s)J_v(rs)\,\mathrm{d}s,$$

which should, of course, be written as $f(r) = \mathscr{H}_v[\tilde{f}(s); s \rightarrow r]$.

The basic results that we need in the following are (Sneddon, 1972; Gladwell, 1980)

$$\mathscr{H}_v\left[r^{v-1}\frac{\partial}{\partial r}\left\{r^{1-v}f(r)\right\}, r \rightarrow s\right] = -s\mathscr{H}_{v-1}\left[f(r); r \rightarrow s\right],$$

$$\mathscr{H}_v\left[\mathscr{B}_v f(r); r \rightarrow s\right] = -s^2\mathscr{H}_v\left[f(r); r \rightarrow s\right],$$

where the differential operator $\mathscr{B}_v$ is given by the formula

$$\mathscr{B}_v = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{v^2}{r^2}.$$

2. Formulation of the problem

Consider a cylindrical coordinate system (r, θ, z) such that the half-space occupies the region $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z < \infty$. Assume that a circular punch of radius a is bonded to the surface of the half-space such that the contact region is given by the relations $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $z = 0$. The contact between the punch and the half-space is perturbed by some torsional forces in the interior of the half-space, in the plane $z = h$, which we model by a distribution of body forces $T(r, z)$ acting in the plane $z = h$ in the tangential direction.

The equilibrium of the medium is governed by the following partial differential equation:

$$\left(\nabla^2 - \frac{1}{r^2}\right)u_\theta + \frac{T}{\mu} = 0, \tag{1}$$

where $\nabla^2 = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2$ is the axisymmetric Laplacian, $u_\theta(r, z)$ is the tangential component of the displacement vector, and μ is the shear modulus of the material of the elastic half-space.

The only non-zero components of the stress tensor are $\sigma_{r\theta}$ and $\sigma_{z\theta}$ which are related to the displacement component u_θ by the following equations:

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \sigma_{z\theta} = \mu\frac{\partial u_\theta}{\partial z}. \tag{2}$$

Under the above assumptions, the problem reduces to that of solving eqn (1) subject to the following boundary conditions:

$$u_\theta(r, 0) = 0, \quad 0 \leqslant r < a,$$

$$\sigma_{z\theta}(r, 0) = 0, \quad r > a. \tag{3}$$

Within the scope of linear elasticity, the solution of the problem can be found by superposing the solutions of the following two simpler problems:

Problem 1. (The Unperturbed Problem): It is required to find the solution of the equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = -\frac{T}{\mu} \tag{4}$$

subject to the following Neumann-type boundary condition:

$$\sigma_{z\theta}(r, 0) = 0, \quad 0 \leqslant r < \infty. \tag{5}$$

Problem 2. (The Perturbation Problem): We need to find the corrective solution which consists of finding the solution of the equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{6}$$

subject to the following mixed boundary conditions:

$$u_\theta(r, 0) = -f(r), \quad 0 \leqslant r < a,$$

$$\sigma_{z\theta}(r, 0) = 0, \quad r > a, \tag{7}$$

where $f(r) = u_\theta^0(r, 0)$ where $u_\theta^0(r, 0)$ is the solution of the unperturbed problem. Here and in the following the superscript zero is used to indicate that the quantity belongs to the unperturbed problem.

In addition, solutions of both problems must satisfy the conditions at infinity to ensure that the elastic field decays as $(r^2 + z^2)^{1/2} \to \infty$.

3. Derivation of the Green's function

In order to solve both problems, we construct the Green's function $G(r, z; r_0, z_0)$ which corresponds to the tangential displacement, $u_\theta(r, z)$, of a stress-free half-space which is subjected to the action of a unit concentrated force uniformly distributed in the tangential direction along a circular ring of radius r_0 in the plane $z = z_0$. Thus, the distribution of body forces equivalent to this case is given by

$$T(r, z) = \frac{1}{2\pi r}\delta(r - r_0)\delta(z - z_0), \tag{8}$$

where $\delta(\cdots)$ is the Dirac delta function.

Using the first order Hankel transformation to eqn (1) with (8), we then have the following boundary value problem:

$$\frac{\mathrm{d}^2\widetilde{G}^1}{\mathrm{d}z^2} - s^2\tilde{G}^1 = \frac{-1}{2\pi\mu}J_1(r_0 s)\delta(z - z_0),$$

$$\mu\frac{\mathrm{d}\widetilde{G}^1}{\mathrm{d}z}\bigg|_{z=0} = 0 \text{ or } \frac{\mathrm{d}\widetilde{G}^1}{\mathrm{d}z}\bigg|_{z} = 0, \quad \widetilde{G}^1(s, z; r_0, z_0) \to 0 \text{ as } z \to \infty. \tag{9}$$

We note that the solution of the homogeneous form of the eqn (9), is given by

$$\widetilde{G}^1(s, z; r_0, z_0) = A\,\mathrm{e}^{-sz} + B\,\mathrm{e}^{sz}. \tag{10}$$

It is easy to verify that this homogeneous solution satisfies the boundary and regularity conditions, if and only if $\widetilde{G^1}(s, z; r_0, z_0) \equiv 0$, which means that the Green's function for the boundary value problem (9) exists (Tricomi, 1985), for which we formally write

$$\widetilde{G^1}(s, z; r_0, z_0) = \left\{ \begin{array}{ll} A_1\,\mathrm{e}^{-sz} + A_2\,\mathrm{e}^{sz}, & 0 \leqslant z < z_0 \\ B_1\,\mathrm{e}^{-sz} + B_2\,\mathrm{e}^{sz}, & z_0 < z < \infty \end{array} \right\}. \tag{11}$$

The function $\tilde{G}^1$ must be continuous at $z = z_0$. This yields the following equation

$$(B_1 - A_1)\,\mathrm{e}^{-sz_0} + (B_2 - A_2)\,\mathrm{e}^{sz_0} = 0. \tag{12}$$

The jump discontinuity of the function $\partial\widetilde{G^1}/\partial z$ across the plane $z = z_0$ is equal to $(-1/2\pi\mu)J_1(r_0 s)$; this gives the equation

$$(B_2 - A_2)\,\mathrm{e}^{sz_0} - (B_1 - A_1)\,\mathrm{e}^{-sz_0} = \frac{-1}{2\pi\mu s} J_1(r_0 s). \tag{13}$$

Denoting $c_i = B_i - A_i$ $(i = 1, 2)$, we rewrite eqns (12) and (13) in the form:

$$c_1\,\mathrm{e}^{-sz_0} + c_2\,\mathrm{e}^{sz_0} = 0,$$

$$c_1\,\mathrm{e}^{-sz_0} - c_2\,\mathrm{e}^{sz_0} = \frac{1}{2\pi\mu s} J_1(r_0 s). \tag{14}$$

Now, using the boundary and regularity conditions [see eqns (9)], we get two more equations, namely,

$$B_2 = 0, \quad A_1 = A_2. \tag{15}$$

Solving eqns (14) and (15), we get

$$A_1 = A_2 = \frac{J_1(r_0 s)}{4\pi\mu s}\,\mathrm{e}^{-sz_0}, \quad B_1 = \frac{J_1(r_0 s)}{2\pi\mu s}\cosh(sz_0), \quad B_2 = 0. \tag{16}$$

Therefore, the Green's function in the Hankel transform domain is given by the formula

$$\widetilde{G^1}(s, z; r_0, z_0) = \frac{1}{4\pi\mu s}\left[\mathrm{e}^{-s|z-z_0|} + \mathrm{e}^{-s(z+z_0)}\right] J_1(r_0 s). \tag{17}$$

Applying Hankel's inverse transformation to eqn (17), we obtain

$$G(r, z; r_0, z_0) = \int_0^\infty s\,\widetilde{G^1}(s, z; r_0, z_0) J_1(rs)\,\mathrm{d}s. \tag{18}$$

Putting eqn (17) into eqn (18), we have

$$G(r, z; r_0, z_0) = \frac{1}{4\pi\mu}\int_0^\infty \left[\mathrm{e}^{-s|z-z_0|} + \mathrm{e}^{-s(z+z_0)}\right] J_1(r_0 s) J_1(rs)\,\mathrm{d}s. \tag{19}$$

For the product of Bessel functions in eqn (19), we use Neumann's addition theorem (Webster, 1924; Rahman, 1997):

$$J_1(rs)J_1(r_0 s) = \frac{1}{\pi}\int_0^\pi \cos\varphi J_0(Rs)\,\mathrm{d}\varphi, \tag{20}$$

where $R = (r^2 + r_0^2 - 2rr_0 \cos\varphi)^{1/2}$, so that the eqn (19) takes the form

$$G(r, z; r_0, z_0) = \frac{1}{4\pi^2\mu}\int_0^{\pi} \cos\varphi\big[S(R, |z - z_0|) + S(R, z + z_0)\big]\,\mathrm{d}\varphi, \tag{21}$$

where the following notation is introduced:

$$S(r, z) = \int_0^{\infty} \mathrm{e}^{-sz} J_0(rs)\,\mathrm{d}s. \tag{22}$$

Closed-form expression for (22) is obtained from Gradshteyn and Ryzhik (1994) as

$$S(r, z) = \frac{1}{R_0},\ \ R_0 = \left(r^2 + z^2\right)^{1/2}. \tag{23}$$

Thus, the expression for the Green's function (21) reduces to

$$G(r, z; r_0, z_0) = \frac{1}{4\pi^2\mu}\int_0^{\pi} \cos\varphi\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\mathrm{d}\varphi, \tag{24}$$

where

$$R_1 = [r^2 + r_0^2 + (z - z_0)^2 - 2rr_0\cos\varphi]^{1/2},\ \ R_2 = [r^2 + r_0^2 + (z + z_0)^2 - 2rr_0\cos\varphi]^{1/2}.$$

Integral (24) is evaluated in closed form (Gradshteyn and Ryzhik, 1994), yielding the following expression for the Green's function:

$$\begin{aligned} G(r, z; r_0, z_0) = \frac{1}{4\pi^2\mu r r_0}\Bigg[& -\frac{l_3^2}{l_1}\Pi\left(\frac{\pi}{2}, p_1^2, p_1\right) + \frac{r^2 + r_0^2 + (z - z_0)^2}{l_1}K(p_1) \\ & -\frac{l_3^2}{l_2}\Pi\left(\frac{\pi}{2}, p_2^2, p_2\right) + \frac{r^2 + r_0^2 + (z + z_0)^2}{l_2}K(p_2)\Bigg], \end{aligned} \tag{25}$$

where $K(\cdots)$ and $\Pi(\cdots)$ are the complete elliptic integrals of the first and third kinds, respectively, and

$$l_1 = [(r + r_0)^2 + (z - z_0)^2]^{1/2},\ \ l_2 = [(r + r_0)^2 + (z + z_0)^2]^{1/2},\ \ l_3 = [(r - r_0)^2 + (z - z_0)^2]^{1/2}$$

$$l_4 = [(r - r_0) + (z + z_0)^2]^{1/2},\ \ p_1 = \frac{2(rr_0)^{1/2}}{l_1},\ \ p_2 = \frac{2(rr_0)^{1/2}}{l_2}.$$

The following relationship holds between Π and E (Gradshteyn and Ryzhik, 1994):

$$\Pi\left(\frac{\pi}{2}, p^2, p\right) = \frac{E(p)}{1 - p^2}, \tag{26}$$

where $E(\cdots)$ is the complete elliptic integral of the second kind.

Substituting into eqn (25), after some simple transformations, we get the following expression for the Green's function:

$$G(r, z; r_0, z_0) = \frac{1}{4\pi^2\mu r r_0}\sum_{j=1}^{2}\left[\frac{r^2 + r_0^2 + \left\{z + (-1)^j z_0\right\}^2}{l_j}K(p_j) - l_j E(p_j)\right]. \tag{27}$$

To the best of our knowledge, the Green's function (27) is new.

Putting $z = 0$ into eqn (27), we obtain the surface Green's function, namely,

$$G(r, 0; r_0, z_0) = \frac{1}{2\pi^2 \mu r r_0}\left[\frac{r^2 + r_0^2 + z_0^2}{l_0}K(p) - l_0 E(p)\right], \tag{28}$$

where

$$l_0 = \left[(r + r_0)^2 + z_0^2\right]^{1/2}, \quad p = \frac{2(rr_0)^{1/2}}{l_0}. \tag{29}$$

4. The solution of the unperturbed problem

With the Green's function (27), it is now straightforward to give the solution of the unperturbed problem; we have

$$u_\theta(r, z) = 2\pi \int_0^\infty r_0 \, \mathrm{d}r_0 \int_0^\infty G(r, z; r_0, z_0) T(r_0, z_0) \, \mathrm{d}z_0. \tag{30}$$

Integrals in (30) should, of course, be understood in the sense of generalized functions or distributions.

However, in order to solve the perturbation problem, we need expression for $u_\theta(r, 0)$. Putting $z = 0$ into eqn (30), we obtain

$$u_\theta(r, 0) = \frac{1}{\pi\mu r}\int_0^\infty \mathrm{d}r_0 \int_0^\infty \left[\frac{r^2 + r_0^2 + z_0^2}{l_0}K(p) - l_0 E(p)\right] T(r_0, z_0) \, \mathrm{d}z_0. \tag{31}$$

Expression (31) allows us to proceed to solve the perturbation problem. However, prior to this, let us illustrate the use of the formula (31) and some of the results derived in Section 3 by considering some specific examples.

Example 1: Consider the case of a concentrated moment of magnitude T_0 acting in the plane $z = h$ in the anticlockwise direction. The body force distribution equivalent to this case is given by the relation:

$$T(r, z) = \frac{-T_0}{2\pi r}\frac{\partial \delta(r)}{\partial r}\delta(z - h). \tag{32}$$

Putting eqn (32) into eqn (30), we obtain

$$u_\theta(r, z) = \frac{T_0 r}{8\pi\mu}\left[\frac{1}{\left\{r^2 + (z - h)^2\right\}^{3/2}} + \frac{1}{\left\{r^2 + (z + h)^2\right\}^{3/2}}\right]. \tag{33}$$

In deriving eqn (33), use has been made of the following property of the Dirac delta function (Zemanian, 1965):

$$\int_b^a f(x)\frac{\partial}{\partial x}\delta(x - x_0) \, \mathrm{d}x = -f'(x_0),$$

and the rules for differentiation and some properties of the elliptic integrals of the first and second kinds.

The solution corresponding to the case where a concentrated moment acts directly on the surface of the half-space can be deduced by letting $h \to 0$ in (33), namely,

$$u_\theta(r, z) = \frac{T_0 r}{4\pi\mu R^3}, \quad R = \left(r^2 + z^2\right)^{1/2}. \tag{34}$$

Solution (34) is consistent with Lur'e (1970) and Chowdhury (1983) obtained by different methods.

The surface displacement of the half-space due to the concentrated moment (32) is obtained by putting $z = 0$ into eqn (33), viz

$$u_\theta(r, 0) = \frac{T_0 r}{4\pi\mu R_h^3}; \quad R_h = \left(r^2 + h^2\right)^{1/2}. \tag{35}$$

Example 2: Consider the case where the elastic half-space is subjected to tangential loads varying linearly with the radius, in the plane $z = h$, i.e.

$$T(r, z) = \tau \frac{r}{a} H(a - r)\delta(z - h), \tag{36}$$

where $H(z - r)$ is the Heaviside step function.

Putting the expression (36) into eqn (31), we obtain

$$u_\theta(r, 0) = \frac{\tau}{\pi\mu a r}\int_0^a r_0\left[\frac{r^2 + r_0^2 + (z-h)^2}{l_h}K(p_h) - l_h E(p_h)\right] \mathrm{d}r_0, \tag{37}$$

where

$$l_h = \left[(r + r_0)^2 + h^2\right]^{1/2}, \; p_h = \frac{2(rr_0)^{1/2}}{l_h}. \tag{38}$$

Similar expressions can be found for any axisymmetrical distribution of internal forces in the half-space.

5. The corrective solution

The surface Green's function (28) allows an elegant formulation of the perturbation problem, i.e. to find the corrective solution. In particular, we note that the surface displacement of the half-space due to an arbitrary distribution of shearing stresses $\sigma_{z\theta}(r, 0) = \tau(r)$ on the half-space over a circular region of radius a can be obtained by putting $T(r, z) = -\tau(r)\delta(z)$ into eqn (31) with the result

$$u_\theta(r, 0) = \frac{-1}{\pi\mu r}\int_0^a\left[\frac{r^2 + r_0^2}{r + r_0}K(q) - (r + r_0)E(q)\right]\tau(r_0)\,\mathrm{d}r_0, \tag{39}$$

where $q = 2(rr_0)^{1/2}/(r + r_0)$.

So, using the first of the boundary conditions in eqn (7) and eqn (39), we observe that the perturbation problem can be reduced to the following integral equation for the unknown contact stresses $\tau(r)$:

$$\int_0^a\left[\frac{r^2 + r_0^2}{r + r_0}K(q) - (r + r_0)E(q)\right]\tau(r_0)\,\mathrm{d}r_0 = \pi\mu r f(r), \; 0 \leqslant r < a. \tag{40}$$

We now develop a method to solve the integral eqn (40), which is based on reducing it to an Abel integral equation using some properties of the complete elliptic integrals of the first and second kinds.

We rewrite eqn (40) in the following form:

$$\begin{aligned}&\int_0^r\left[\frac{r\{1+(r_0^2/r^2)\}}{1+(r_0/r)}K\left(\frac{2(r_0/r)^{1/2}}{1+(r_0/r)}\right)-r\left(1+\frac{r_0}{r}\right)E\left(\frac{2(r_0/r)^{1/2}}{1+(r_0/r)}\right)\right]\tau(r_0)\,\mathrm{d}r_0\\&\quad+\int_r^a\left[\frac{r_0\{1+(r^2/r_0^2)\}}{1+(r/r_0)}K\left(\frac{2(r/r_0)^{1/2}}{1+(r/r_0)}\right)-r_0\left(1+\frac{r}{r_0}\right)E\left(\frac{2(r/r_0)^{1/2}}{1+(r/r_0)}\right)\right]\tau(r_0)\,\mathrm{d}r_0=\pi\mu rf(r).\end{aligned}\tag{41}$$

Using Landen transformations for the complete elliptic integrals of the first and second kinds (Gradshteyn and Ryzhik, 1994), it can be shown that the following relations hold:

$$\frac{1}{1+k}K\left(\frac{2k^{1/2}}{1+k}\right)=K(k);\quad(1+k)E\left(\frac{2k^{1/2}}{1+k}\right)=2E(k)-(1-k^2)K(k).\tag{42}$$

In view of the relations (42), eqn (41) takes the form

$$\begin{aligned}&\int_0^r\left[r\left(1+\frac{r_0^2}{r^2}\right)K\left(\frac{r_0}{r}\right)-2rE\left(\frac{r_0}{r}\right)+\frac{r^2-r_0^2}{r}K\left(\frac{r_0}{r}\right)\right]\tau(r_0)\,\mathrm{d}r_0\\&\quad+\int_r^a\left[r_0\left(1+\frac{r^2}{r_0^2}\right)K\left(\frac{r}{r_0}\right)-2r_0E\left(\frac{r}{r_0}\right)+\frac{r_0^2-r^2}{r_0}K\left(\frac{r}{r_0}\right)\right]\tau(r_0)\,\mathrm{d}r_0=\pi\mu rf(r).\end{aligned}\tag{43}$$

We further transform eqn (43) using the following integral representations for the complete elliptic integrals (Gradshteyn and Ryzhik, 1994):

$$K\left(\frac{r_0}{r}\right)=r\int_0^{r_0}\frac{\mathrm{d}s}{(r_0^2-s^2)^{1/2}(r^2-s^2)^{1/2}};\ E\left(\frac{r_0}{r}\right)=\frac{1}{r}\int_0^{r_0}\left(\frac{r^2-s^2}{r_0^2-s^2}\right)^{1/2}\mathrm{d}s.\tag{44}$$

Expressions for $K(r/r_0)$ and $E(r/r_0)$ can be obtained by interchanging the positions of r and r_0 in eqns (44).

Putting (44) into eqn (43), after some simple manipulations, we obtain

$$\int_0^r\tau(r_0)\,\mathrm{d}r_0\int_0^{r_0}\frac{s^2}{(r^2-s^2)^{1/2}(r_0^2-s^2)^{1/2}}\,\mathrm{d}s+\int_r^a\tau(r_0)\,\mathrm{d}r_0\int_0^{r}\frac{s^2}{(r^2-s^2)^{1/2}(r_0^2-s^2)^{1/2}}\,\mathrm{d}s=\frac{\pi\mu}{2}rf(r).\tag{45}$$

The expression on the left can be represented as a double integral over the trapezoid bounded by the lines $s=0$, $s=r_0$, $s=r$ and $r_0=a$. Changing the order of integration in this integral, we obtain

$$\frac{2}{\pi}\int_0^r\frac{s^2}{(r^2-s^2)^{1/2}}\,\mathrm{d}s\int_s^a\frac{\tau(r_0)}{(r_0^2-s^2)^{1/2}}\,\mathrm{d}r_0=\mu rf(r).\tag{46}$$

If we write

$$\int_s^a\frac{\tau(r_0)}{(r_0^2-s^2)^{1/2}}\,\mathrm{d}r_0=\frac{\psi(s)}{s^2},\ 0\leqslant s\leqslant a,\tag{47}$$

then eqn (46) reduces to Schlömilch integral equation

$$\frac{2}{\pi}\int_0^r \frac{\psi(s)}{(r^2-s^2)^{1/2}}\,\mathrm{d}s = \mu r f(r),\ \ 0 \leqslant r \leqslant a, \tag{48}$$

whose solution is (Lebedev et al., 1965; Tricomi, 1985)

$$\psi(s) = \mu\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \frac{t^2 f(t)}{(s^2-t^2)^{1/2}}\,\mathrm{d}t. \tag{49}$$

On the other hand, the integral eqn (47) is that of Abel, whose solution is given by the formula (Lebedev et al., 1965; Tricomi, 1985)

$$\tau(r) = \frac{-2}{\pi}\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\psi(s)\,\mathrm{d}s}{s(s^2-r^2)^{1/2}}. \tag{50}$$

Eqns (49) and (50) give the contact stresses under the punch, thus formally completing the solution of the titled problem.

We now find the asymptotic behaviour of the contact stresses $\tau(r)$ in the sense of Erdelyi as the punch edge is approached ($r \to a_-$), that is to find the first term in the asymptotic expansions of the expression (50) as $r \to a_-$; we find that

$$\tau(r) \approx \left(\frac{2}{a}\right)^{1/2}\frac{\psi(a)}{\pi(a-r)^{1/2}} = \left(\frac{2}{a}\right)^{1/2}\frac{\mu}{\pi a(a-r)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{t^2 f(t)\mathrm{d}t}{(a^2-t^2)^{1/2}} \text{ as } r \to a_-. \tag{51}$$

We observe that the contact stresses under the punch exhibit square root singularity as the punch edge is approached. The stress intensity factor K near the rim of the punch is given by the expression

$$K = \lim_{r\to a_-}(a-r)^{1/2}\left[\sigma^0_{z\theta}(r,0)|_{0\leqslant r<a} + \tau(r)\right] = \lim_{r\to a_-}(a-r)^{1/2}\tau(r) = \left(\frac{2}{a}\right)^{1/2}\frac{\mu}{\pi a}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{t^2 f(t)\,\mathrm{d}t}{(a^2-t^2)^{1/2}}. \tag{52}$$

Formula (52) allows us to find the stress intensity factor near the rim of the stamp for any arbitrary axisymmetric buried torsional force acting inside an isotropic elastic half-space, which can be used, in conjunction with a failure criterion, to determine the condition of crack initiation and crack propagation near the edge of the punch.

We now turn to determine the integral characteristic of the problem. The total moment necessary to oppose the rotation of the punch is

$$M = -2\pi\int_0^a r^2\left[\sigma^0_{z\theta}(r,0)|_{0\leqslant r<a} + \tau(r)\right]\mathrm{d}r = -2\pi\int_0^a r^2\tau(r)\,\mathrm{d}r. \tag{53}$$

Putting (50) into eqn (53), we obtain

$$M = 4\int_0^a r^2\left[\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\psi(s)\,\mathrm{d}s}{s(s^2-r^2)^{1/2}}\right]\mathrm{d}r = -8\int_0^a \psi(s)\,\mathrm{d}s = -8\mu\int_0^a \frac{t^2 f(t)\,\mathrm{d}t}{(a^2-t^2)^{1/2}}. \tag{54}$$

We now proceed to consider some specific cases of buried torsional loading.

Example 1: Consider the case where the bonded contact is perturbed by a concentrated moment of magnitude T_0 acting in the plane $z = h$ in the anticlockwise direction. In this case, the function $f(r)$ is given by the expression (35). Substituting into (54), we obtain

$$M = \frac{-2T_0 p^3}{\pi} \int_0^{\pi/2} \frac{\sin^3 \theta \, d\theta}{\left(1 + p^2 \sin^2 \theta\right)^{3/2}} = \frac{2T_0 p^2}{\pi} \frac{dI(p)}{dp}, \tag{55}$$

where $p = 1/H$, $H = h/a$ and

$$I(p) = \int_0^{\pi/2} \frac{\sin \theta \, d\theta}{\left(1 + p^2 \sin^2 \theta\right)^{1/2}}. \tag{56}$$

Closed-form expression for the integral (56) is found in Gradshteyn and Ryzhik (1994) as

$$I(p) = \frac{\tan^{-1} p}{p}. \tag{57}$$

Insertion of (57) into (55) then yields

$$M = \frac{2T_0}{\pi} \left[\frac{H}{1 + H^2} - \tan^{-1} \frac{1}{H} \right]. \tag{58}$$

Example 2: Consider the case where the bonded contact is disturbed by a concentrated force P uniformly distributed along a circular ring of radius b in the tangential direction in the anticlockwise orientation. In this case,

$$T(r, z) = \frac{P}{2\pi r} \delta(r - b) \delta(z - h). \tag{59}$$

Putting (59) into (31), we obtain

$$f(r) = PG(r, 0; b, h) = \frac{P}{2\pi^2 \mu r b} \left[\frac{r^2 + b^2 + h^2}{l_b} K(p_b) - l_b E(p_b) \right], \tag{60}$$

where

$$l_b = \left[(r + b)^2 + h^2 \right]^{1/2}, \quad p_b = \frac{2(rb)^{1/2}}{l_b}.$$

Putting (60) into (54), we obtain

$$M = \frac{-4Pa}{\pi^2 B} \int_0^{\pi/2} \sin \theta \left[\frac{\sin^2 \theta + B^2 + H^2}{\Gamma} K(\Omega) - \Gamma E(\Omega) \right] d\theta, \tag{61}$$

where $B = b/a$ and

$$\Gamma = \left[(\sin \theta + B)^2 + H^2 \right]^{1/2}, \quad \Omega = \frac{2(B \sin \theta)^{1/2}}{\Gamma}.$$

Example 3: Consider the case where the elastic half-space is subjected to tangential loads τ varying linearly with the radius of the punch, in the plane $z = h$, the direction of the load being in the anticlockwise direction. In this case, the function $f(r)$ is given by eqn (37), which upon substitution into (54) yields

$$M = \frac{-8\tau a^3}{\pi} \int_0^{\pi/2} \sin\theta \, d\theta \int_0^1 x \left[\frac{\sin^2\theta + x^2 + H^2}{\tilde{\Gamma}} K(\tilde{\Omega}) - \tilde{\Gamma} E(\tilde{\Omega}) \right] dx, \tag{62}$$

where

$$\tilde{\Gamma} = \left[(\sin\theta + x)^2 + H^2 \right]^{1/2}, \quad \tilde{\Omega} = \frac{2(x \sin\theta)^{1/2}}{\tilde{\Gamma}}.$$

Expressions similar to eqns (58), (61) and (62) can be derived for any axisymmetrical distribution of buried torsional forces and the prowess of the present approach lies in this generality.

6. Generalizations to transversely isotropic solids

The results obtained in Section 5 can be easily extended to the case where the half-space is made up of a transversely isotropic material. The equation of equilibrium governing the pure torsion of a transversely isotropic material is given by (Kassir and Sih, 1975)

$$\left(\frac{c_{11} - c_{12}}{2} \right) \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right) + c_{44} \frac{\partial^2 u_\theta}{\partial z^2} + T = 0, \tag{63}$$

where c_{11}, c_{12} and c_{44} are three of the five independent elastic constants of the transversely isotropic material. Numerical values of these constants for some transversely isotropic materials are given in (Kassir and Sih, 1975).

The non-zero components of the stress tensor are related to the tangential displacement component by the equations:

$$\sigma_{z\theta} = c_{44} \frac{\partial u_\theta}{\partial z}, \quad \sigma_{\theta r} = \left(\frac{c_{11} - c_{12}}{2} \right) \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right). \tag{64}$$

Equation (63) can be rewritten as

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z_1^2} + \frac{T}{\mu^*} = 0, \tag{65}$$

where

$$z_1 = \Lambda z, \ \mu^* = \frac{c_{11} - c_{12}}{2}, \ \Lambda = \left(\frac{\mu^*}{c_{44}} \right)^{1/2}. \tag{66}$$

We observe that with the scaling factors (66), eqn (65) exactly resembles eqn (1) and hence solution for this case would be similar to what is presented in Sections 3, 4 and 5. Therefore, without repeating the same solution procedure, we simply list the key results:

$$G(r, z; r_0, z_0) = \frac{1}{4\pi^2 r r_0 \beta} \sum_{j=1}^{2} \left[\frac{r^2 + r_0^2 + \Lambda^2 \{ z + (-1)^j z_0 \}^2}{\lambda_j} K(s_j) - \lambda_j E(s_j) \right], \tag{67}$$

where

$$\beta = \mu^* \Lambda, \quad \lambda_1 = \left[(r + r_0)^2 + \Lambda^2 (z - z_0)^2 \right]^{1/2},$$

$$\lambda_2 = \left[(r + r_0)^2 + \Lambda^2 (z + z_0)^2 \right]^{1/2}, \quad s_j = \frac{2(rr_0)^{1/2}}{\lambda_j}. \tag{68}$$

The result corresponding to eqn (33) assumes the form:

$$u_\theta(r, z) = \frac{T_0 r}{4\pi\beta} \left[\frac{1}{\left\{ r^2 + \Lambda^2 (z - h)^2 \right\}^{3/2}} + \frac{1}{\left\{ r^2 + \Lambda^2 (z + h)^2 \right\}^{3/2}} \right]. \tag{69}$$

Eqn (69) gives the displacement field in a transversely isotropic half-space caused by the action of a concentrated moment in its interior in the plane $z = h$, the direction of the moment being anticlockwise.

Expression for the moment M can be obtained by replacing μ by β. The results corresponding to eqns (58), (61) and (62) can be deduced by simply replacing H by ΛH.

7. Closure

In the present article, we have presented an extension of Reissner–Sagoci's classical solution to the problem of bonded contact of a rigid circular punch with a homogeneous, elastic isotropic half-space, which is under any arbitrary axisymmetrical distribution of buried torsional forces. Specific examples have been considered. Furthermore, generalizations of these results have been given for a transversely isotropic half-space. The method of solution developed here can be adapted to the investigation of other mathematically similar mixed boundary value problems of elasticity.

Acknowledgements

The writer is indebted to an anonymous reviewer for his/her kind criticisms on an earlier version of the manuscript.

References

Chowdhury, K.L., 1983. Solution of the problem of a concentrated torque on a semi-infinite space by similarity transform. Journal of Elasticity 13, 87–90.

De Pater, A.D., Kalker, J.J. (Eds.), 1974. The Mechanics of Contact Between Deformable Bodies. Proc. I.U.T.A.M. Symp. Enschede. Delft University Press.

Fabrikant, V.I., Selvadurai, A.P.S., Xistris, G.D., 1985. Asymmetric problem of loading under a smooth punch. ASME Journal of Applied Mechanics 52, 681–685.

Galin, L.A. (Ed.), 1976. Development of The Theory of Contact Problems in The USSR. Nauka, Moscow (in Russian).

Gladwell, G.M.L., 1980. Contact Problems in The Classical Theory of Elasticity. Sijhoff andNoordhoff, Alphen aan den Rijn, The Netherlands.

Gradshteyn, I.S., Ryzhik, I.M., 1994. Table of Integrals, Series and Products. Academic Press, New York.

Hanson, M.T., Puja, I.W., 1997. The Reissner–Sagoci problem for the transversely isotropic half-space. ASME Journal of Applied Mechanics 64, 692–694.

Hills, D.A., Nowell, D., Sackfield, A., 1993. Mechanics of Elastic Contacts. Butterworth-Heineman, Philadelphia.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press.

Kikuchi, N., Oden, J.T., 1988. Contact Problems in Elasticity: A Survey of Variational Inequalities and Finite Element Methods. SIAM, Philadelphia.

Kassir, M.K., Sih, G.C., 1975. Three-Dimensional Crack Problems. Noordhoff, Leyden.

Lebedev, N.N., Skalskaya, I.P., Ufliand, Ya. S., 1965. Problems of Mathematical Physics. Prentice-Hall, New Jersey.

Lur̕e, A.I., 1970. Theory of Elasticity. Nauka, Moscow (in Russian).

Rahman, M., 1997. On a generalization of Neumann's formula for the product of two first kind Bessel functions of integral orders. Journal of Applied Mathematics and Mechanics (ZAMM) 77(2), 155–156.

Selvadurai, A.P.S., 1978. The interaction between a rigid circular punch on an elastic half-space and a Mindlin force. Mechanics Research Communications 5 (2), 57–64.

Selvadurai, A.P.S., 1981. Fundamental results concerning the settlement of a rigid foundation due to an inclined anchor load applied at the interior of the soil medium. International Journal for Numerical and Analytical Methods in Geomechanics 5, 209–223.

Selvadurai, A.P.S., 1982. The statical Reissner–Sagoci problem for an internally loaded transversely isotropic half-space. International Journal of Engineering Science 20 (12), 1365–1372.

Selvaduri, A.P.S., 1990. Hertzian contact in the presence of a Mindlin force. Zeitschrift fur angewandte Mathematik und Physik (ZAMP) 41, 865–874.

Sneddon, I.N., 1946. Note on a boundary value problem of Reissner and Sagoci. Journal of Applied Physics, pp. 130-132.

Sneddon, I.N., 1951. Fourier Transforms. McGraw-Hill, New York.

Sneddon, I.N., 1972. The Use of Integral Transforms. McGraw-Hill, New York.

Tricomi, F., 1985. Integral Equations. Dover, New York.

Ufliand, Ya. S., 1965. Survey of Articles on The Application of Integral Transforms in The Theory of Elasticity. Tech. Rep. AFOSR-65-1556, North Carolina State University.

Webster, A.G., 1927. Partial Differential Equations of Mathematical Physics. Stechert, New York.

Zemanian, A.H., 1965. Distribution Theory and Transform Analysis. McGraw-Hill, New York.